Resonant Frequencies of Higher Order Modes in Cylindrical Anisotropic Dielectric Resonators

Michael E. Tobar, *Student Member, IEEE*, and Anthony G. Mann

Abstract—**An improved method is developed which allows the determination of mode frequencies to high accuracy in cylindrical anisotropic dielectric resonators. This is an extension of Garault and Guillon's method from isotropic to anisotropic dielectrics, applied to four different classes of field patterns. The theory is confirmed by room temperature measurements in two sapphire crystals of different aspect ratios, and in cryogenic sapphire resonators used in high stability fixed and tunable oscillators. The sensitivity of mode frequency to dimensional and permittivity perturbations is analyzed.**

I. Introduction

THE HIGHER order modes in cylindrical sapphire dielectric resonators at cryogenic temperatures [1] can exhibit extremely high Q factors ($>10^7$) and can be used to construct ultrastable low noise microwave oscillators [2]–[5]. Other anisotropic dielectric crystals such as rutile [6], [7] and lithium niobate [8] offer higher permittivities but also much higher losses.

A method which calculates the frequency of the lowest order mode in a cylindrical isotropic dielectric [9] has been extended to higher order modes in an anisotropic crystal. Previous equations are shown only to be valid for quasi TE modes with even mode number in the axial direction. Four different axial match equations are derived depending whether they are quasi TE or quasi TM, and have an odd or even axial mode number. A general radial match equation is derived. Combining it with the relevant axial equation forms a set of two coupled transcendental equations which can be solved numerically. This is a general treatment of higher order modes in an anisotropic medium, although previously whispering gallery mode approximations for anisotopic crystals have been used successfully [10], [11].

Theory is applied to two sapphire crystals of different aspect ratios. Very good agreement is found even though the permittivity is only about ten. The anisotropy forces the TM mode families to be lower in frequency than the TE, which explains the discrepancy between theory and experiment of previous work [2]. The theory presented successfully predicts frequency shifts from room to liquid helium temperature due to the change in permittivity, and the tuning range of a tunable sapphire resonator due to the effective change in height.

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II. Theory

Cylindrical anisotropic crystals in free space are analysed relative to the coordinate system defined in Fig. 1, with the c-axis of the crystal parallel to the z axis. The permittivity parallel and perpendicular to the c-axis is defined as ϵ_z and ϵ_r, respectively. Thus $\epsilon_\phi = \epsilon_r$ and we assume no off diagonal terms in the permittivity tensor.

The problem is solved using Maxwell's equations for anisotropic media. Assuming the divergence of E is not dependent on z, then applying separation of variables on the z component of the electromagnetic field, we can write

$$E_{z1} = AJ_m(k_E r)\begin{matrix}\cos(m\phi)\\ \sin(m\phi)\end{matrix}\left(P_1\exp(-j\beta z) + P_2\exp(+j\beta z)\right) \tag{1a}$$

$$E_{z2} = CK_m(k_{\text{out}} r)\begin{matrix}\cos(m\phi)\\ \sin(m\phi)\end{matrix}\left(P_1\exp(-j\beta z) + P_2\exp(+j\beta z)\right) \tag{1b}$$

$$E_{z3} = EJ_m(k_E r)\begin{matrix}\cos(m\phi)\\ \sin(m\phi)\end{matrix}\exp(-\alpha z) \tag{1c}$$

$$H_{z1} = BJ_m(k_H r)\begin{matrix}\sin(m\phi)\\ \cos(m\phi)\end{matrix}\left(P_1\exp(-j\beta z) + P_2\exp(+j\beta z)\right) \tag{1d}$$

$$H_{z2} = DK_m(k_{\text{out}} r)\begin{matrix}\sin(m\phi)\\ \cos(m\phi)\end{matrix}\left(P_1\exp(-j\beta z) + P_2\exp(+j\beta z)\right) \tag{1e}$$

$$H_{z3} = FJ_m(k_H r)\begin{matrix}\sin(m\phi)\\ \cos(m\phi)\end{matrix}\exp(-\alpha z) \tag{1f}$$

where $k_E^2 = \epsilon_z k_0^2 - \beta^2$, $k_H^2 = \epsilon_r k_0^2 - \beta^2$ and $k_{\text{out}}^2 = \beta^2 - k_0^2$. Here m is the azimuthal mode number, β the longitudinal dielectric propagation constant, k_0 the free space

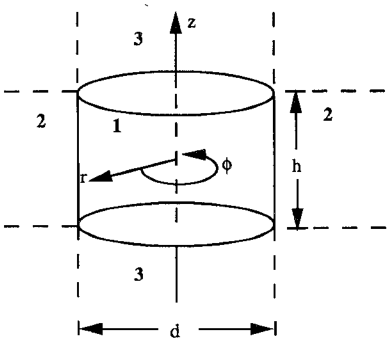

Fig. 1. Open dielectric crystal is analyzed in cylindrical coordinates $\{r, \phi, z\}$. Resonant frequencies are solved by matching tangential fields between regions 1 and 2, and regions 1 and 3.

wave number, k_{out} the radial propagation constant outside the dielectric, and k_E and k_H the radial dielectric propagation constants parallel and perpendicular to the c-axis, respectively. From the z component, Maxwell's equations can then be used to obtain all other electromagnetic field components [12].

A. Radial Match

By matching tangential components of the H and E fields between regions 1 and 2, the following transcendental equation is obtained:

$$\left(\frac{\epsilon_r J_m'(x_E)x_E}{x_H^2 J_m(x_E)} + \frac{K_m'(y)}{yK_m(y)}\right)\left(\frac{J_m'(x_H)}{x_H J_m(x_H)} + \frac{K_m'(y)'}{yK_m(y)}\right) = m^2\frac{(x_H^2 + \epsilon_r y^2)(x_H^2 + y^2)}{x_H^4 y^4} \quad (2)$$

where $x_E = k_E d/2$, $x_H = k_H d/2$ and $y = k_{out} d/2$. For a fixed diameter this equation is a function of two variables, k_0 and β.

In general y becomes imaginary for the lower order whispering gallery-like mode families, where β is small and m is large. In this case the Evanescent Bessel Function becomes a Hankel Function of the second kind [13]. Hence we employ algorithms which allow complex arguments.

B. Axial Match

Field components E_z and H_z must be orthogonal in space and hence cannot coexist with the same dependence on z. Assuming the same dependence however simplifies proceedings by allowing the axial match to be calculated independently of the radial match. To be consistent with the z dependence in (1a)–(1f), quasi TE modes ($E_z \approx 0$) or quasi TM modes ($H_z \approx 0$) must be assumed. Four different transcendental equations are derived by matching tangential fields between regions 1 and 3. The radial and axial mode numbers are n and p,

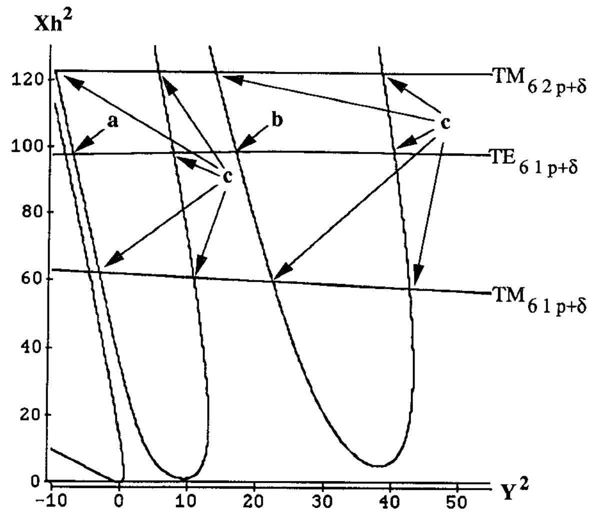

Fig. 2. Intersections of (2) and (3a) are illustrated in $\{Xh^2, Y^2\}$ space. (a) $TE_{61\delta}$ mode. (b) $TE_{612+\delta}$ mode. (c) Spurious solutions.

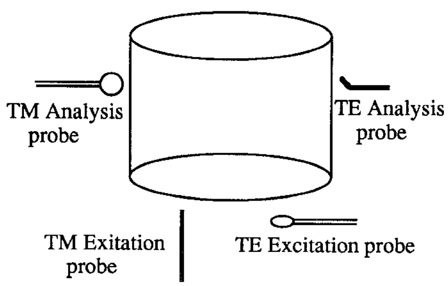

Fig. 3. TM modes are excited by creating an E_z field, while TE modes are excited by creating a H_z field.

respectively:

$$TE_{mnp+\delta} \quad p \text{ even}; \quad k_0^2 = \beta^2\frac{1+\tan^2(\beta h/2)}{\epsilon_r - 1} \quad (3a)$$

$$p \text{ odd}; \quad k_0^2 = \beta^2\frac{1+\cot^2(\beta h/2)}{\epsilon_r - 1} \quad (3b)$$

$$TM_{mnp+\delta} \quad p \text{ even}; \quad k_0^2 = \beta^2\frac{1+(\tan(\beta h/2)/\epsilon_r)^2}{\epsilon_z - 1} \quad (3c)$$

$$p \text{ odd}; \quad k_0^2 = \beta^2\frac{1+(\cot(\beta h/2)/\epsilon_r)^2}{\epsilon_z - 1}. \quad (3d)$$

Equation (3a) is the same as derived by Garault and Guillon [9], which was solved with the isotropic version of Equation (2).

C. Solving the Coupled Equations

Equations (2) and (3) are solved using Mathematica [14]. Solutions are found graphically on a x_H^2 versus y^2 graph, then more accurately using a Newton–Raphson technique. For a given value of m, (2) gives an infinite set of solutions in $\{x_H^2, y^2\}$ space, which are almost perpendicular to the x_H^2 axis. This distinguishes the radial mode number n, and whether they are TE or TM. Equation (3)

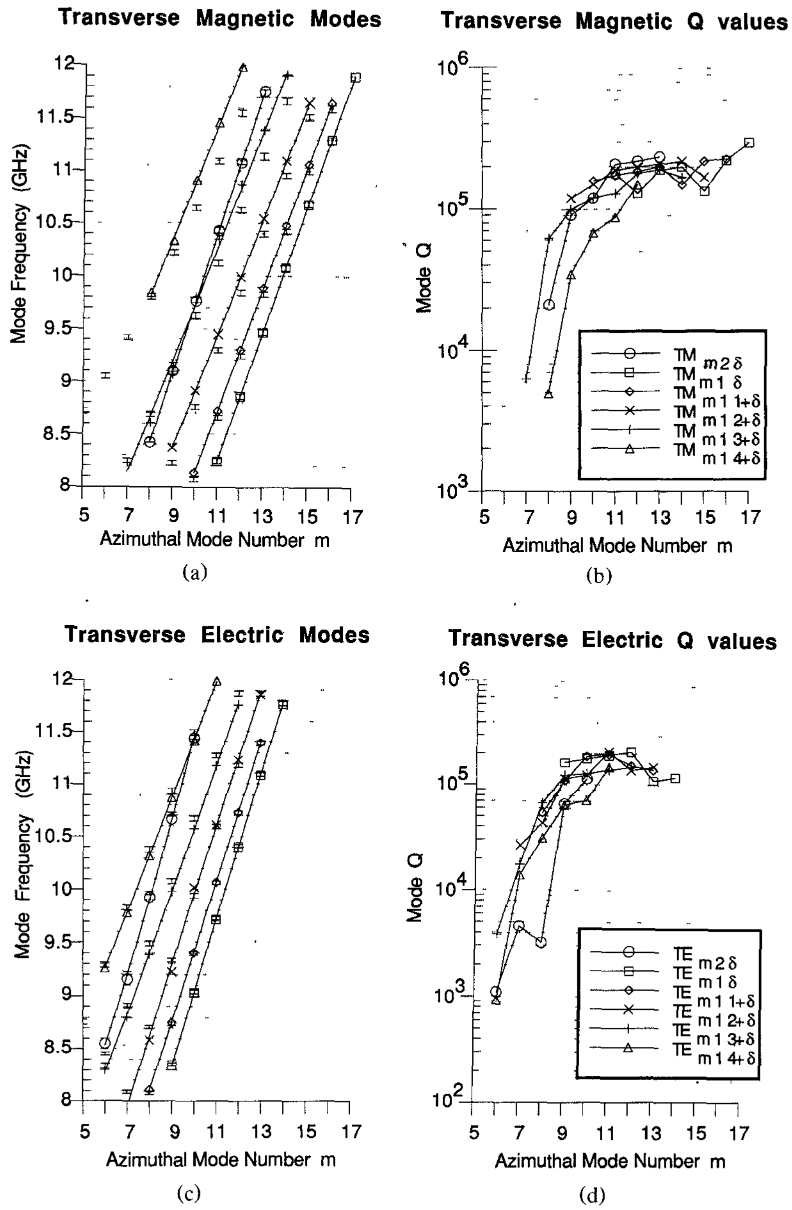

Fig. 4. Mode frequencies and Q values as a function of azimuthal mode number, for a 50.0 mm diameter and 30.0 mm high sapphire crystal. Theoretical points are plotted as error bars due to uncertainties in dimensions and permittivities, while experimental points are joined by lines.

gives an infinite set of solutions nearly parallel to the x_H^2 axis. Mode frequencies are solved from the intersection of these two solution sets. Care must be taken to avoid spurious solutions due to the restriction of p being either even or odd or modes being TE or TM. Fig. 2 illustrates how the solution of the $TE_{61\delta}$ in the smaller sapphire is found graphically.

III. Experimental Verification

A. Cylindrical Oriented Sapphire

Fig. 3 shows how quasi TE and TM modes are distinguished. To excite a TM or TE mode, an E_z or H_z field is excited, respectively. In reality all modes are hybrid, and experimentally one can excite higher order axial mode number families with either a TM or TE probe. Analysis of azimuthal and axial mode numbers is done by observing the H_ϕ or E_ϕ field, respectively.

An experimental study of resonant modes was conducted from 8 to 12 GHz for two cylindrical pieces of sapphire with different aspect ratios. Figs. 4 and 5 compare experimental and theoretical mode frequencies, and observed open resonator Q values. There is better agreement for the whispering gallery type families of low axial mode number p, as they are more TM or TE like. The difference between the calculated and measured mode frequencies generally increases with p and decreases with m. In the 50.0 mm diameter sapphire crystal for modes with $p=0$ the error is less than 0.1%, which is smaller than estimated uncertainties in permittivities and dimensions, while above $p=2$ errors can be of the order 1% in TE modes and 2% in TM modes [15]. In the 31.8 mm

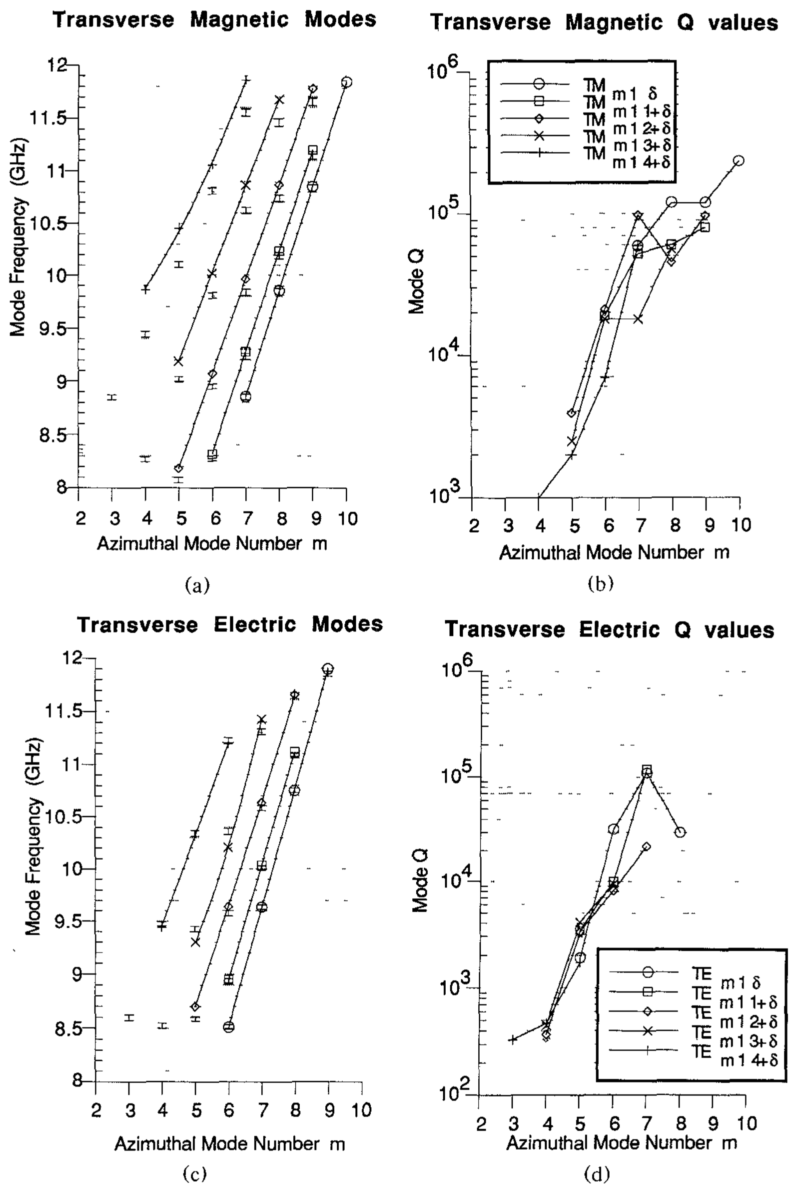

Fig. 5. Mode frequencies and Q values as a function of azimuthal mode number, for a 31.8 mm diameter and 30.2 mm high sapphire crystal. Theoretical points are plotted as error bars due to uncertainties in dimensions and permittivities, while experimental points are joined by lines.

diameter sapphire crystal, the errors are generally larger: for example about 0.2% for $p = 0$ modes.

The permittivity of sapphire above and below X-band has been measured previously with slightly conflicting results [16], [17]. To be consistent with both reports we can be confident that ϵ_z is 11.6245 ± 0.0355 and 11.355 ± 0.015 at 300 K and 4 K, respectively, and ϵ_r is 9.407 ± 0.012 and 9.2895 ± 0.0255 at 300 K and 4 K, respectively.

The power radiated from a given mode of the open dielectric resonator could be calculated from the field components (1) by integrating the Poynting vector over the resonator surface, but is beyond the scope of this paper. Within each mode family the power radiated from the open resonator decreases as the azimuthal mode number increases [10]. The observed open resonator Q thus increases monotonically with m to a limit set by the dielectric loss tangent. At room temperature and 10 GHz this limit is about $2 \cdot 10^5$ and scales inversely with frequency [1]. For the 50 mm diameter sapphire the density of modes in the 11 GHz region is large and interacting modes are common. In this region some mode Q's may actually decrease for an increase in m due to reactive coupling to nearby low Q modes [18].

B. Frequency Sensitivity

The sensitivities of mode frequency to perturbations in permittivities and dimensions ($\Delta\epsilon_r$, $\Delta\epsilon_z$, Δd and Δh) have been calculated for the large and small sapphires. To within the precision of the calculation (0.2 to 2%) the sums of the sensitivities obey the expected relations:

$$\frac{\partial f}{\partial \epsilon_r}\frac{\epsilon_r}{f} + \frac{\partial f}{\partial \epsilon_z}\frac{\epsilon_z}{f} = -\frac{1}{2}$$

and

$$\frac{\partial f}{\partial d}\frac{d}{f}+\frac{\partial f}{\partial h}\frac{h}{f}=-1.$$

General trends are observed in the sensitivities. Within a mode family as azimuthal mode number increases the dependence of mode frequency on diameter increases. For both TE and TM modes with high azimuthal and low axial mode number the frequency depends mainly ($\sim 98\%$) on the diameter. As the axial mode number increases the frequency dependence on the height increases, and becomes comparable with diameter when $p > 3$ for TE modes and $p > 2$ for TM modes. Quasi TE modes have most of their electric field perpendicular to the z-axis and are seen to depend manily on ϵ_r, while quasi TM modes have most of their electric field parallel to the z-axis and depend mainly on ϵ_z. Within a mode family this dependence on the respective dielectric constant increases with azimuthal mode number. For the higher order axial mode families ($p > 3$ for TE modes and $p > 2$ for TM modes) the dependences on ϵ_r and ϵ_z are about equal. Solutions are then in the hybrid regime. This is why solutions are less accurate for these families as (3) assumes pure TE and TM modes while (2) does not.

C. Applications to Sapphire Loaded Superconducting Cavities (SLOSC)

Details of a high stability 9.73 GHz SLOSC oscillator have been presented previously [3], [4]. The operational sapphire resonance can now be identified as $TE_{61\delta}$. Theory predicts a 66 MHz shift in frequency when cooled due to the change in dielectric constant, exactly what is measured.

Details of a tunable SLOSC have been presented previously [18], [19]. It consists of a 3 cm diameter cylindrical sapphire crystal, and an axially driven tuning disc 0.3 cm thick. Tuning ranges can be predicted with presented theory. Modes with higher tuning ranges have larger axial mode numbers. Modes analysed previously [18] at 4 K are now identified as $TE_{61\delta+1}$ at 10.221 GHz and $TM_{81\delta+1}$ at 10.44 GHz. Tuning ranges of these modes are predicted to be 94 and 99 MHz, respectively, which agrees favorably with experiment.

IV. Conclusion

Improved theory for multimode analysis of anisotropic dielectric resonators was presented. This has lead us to a very good understanding of electromagnetic resonances in sapphire crystals, with potential application to design.

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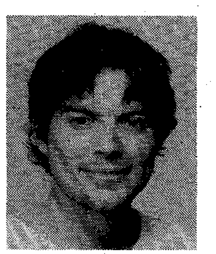

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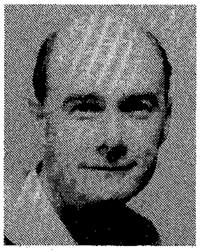

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